USE OF THE HAMILTON – OSTROGRADSKII PRINCIPLE IN PROBLEMS OF THE THEORY OF NONLINEAR OSCILLATIONS

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The Hamilton – Ostrogradskii principle can be used directly to obtain approximate steady polyharmonic oscillations in nonlinear systems. Authors of [1] have shown its use for approximate frequency determinations in nonlinear conservative systems. The following quantities are used in the present paper: mean value of the Lagrangian and the mean value of a function W defined in terms of nonconservative forces. These functions are obtained from a solution whose form is established with the help of some additional considerations. In the end we obtain equations in terms of mean quantities referred to above, written in a very compact form and used to determine the unknown parameters of the solution. As an example, we consider the interaction of harmonics in a system with dry friction.

1. Approximate investigation of oscillations in nonlinear systems usually involves a requirement that the solution should be of a certain form selected a priori. The choice may be based on experimental or theoretical investigations of other similar systems. Thus in [2 to 4] we find that when an external force expressible in terms of a trigonometric series acts on a nonlinear dissipative system, then the resulting motion obeys, as a rule, a law in which harmonics present in the applied force, clearly predominate. This justifies the approach, in which we seek the first approximation to the oscillations of such a system in the form of a sum of these harmonics, which are present in the external force. Having chosen the form of the required solution, we must now determine the unknown parameters entering this solution and this often presents considerable difficulties. Various methods of obtaining the necessary equations exist. We shall illustrate the use of the Galerkin method and of the harmonic balance method in investigating periodic oscillations, together with the methods of linearisation according to the criterion of the minimum mean square deviation and of linearization with respect to the distribution function. The last two methods can also be used for aperiodic oscillations [5]. Alternatively, the Hamilton – Ostrogradskii method can be applied directly and this approach offers, in some cases substantial advantages.

Suppose, for example, that we have a system with n degrees of freedom whose motion can approximately be represented by

$$q_j(t) = \sum_{i=1}^{r} (A_{ij}' \sin \omega_i t + A_{ij}'' \cos \omega_i t) \tag{1.1}$$

In addition we shall assume that the Lagrangian L of this system is known and that the elementary work $\delta' W$ done by the nonconservative generalized forces $Q_j(q_\nu, q_\nu^\cdot, t)$ over the possible displacements $\delta q_j^\cdot$ has been already computed

$$L(q_\nu, q_\nu^\cdot) = T - \Pi, \qquad \delta' W = Q_1 \delta q_1 + \ldots + Q_n \delta q_n$$

For the system under consideration we shall state the Hamilton – Ostrogradskii principle in the form which does not impose the usual constraints on the variations of generalized coordinates at the initial and final moment of time

$$\int_0^\tau (\delta L + \delta' W)\, dt = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}}\, \delta q_j \Big|_{t=0}^{t=\tau} \tag{1.2}$$

We can always assume that the frequencies in (1.1) are fixed (this also applies to free oscillations), therefore, varying the amplitudes only, we obtain

$$\delta q_j = \sum_{i=1}^{r} (\delta A_{ij}' \sin \omega_i t + \delta A_{ij}' \cos \omega_i t) \tag{1.3}$$

$$\delta L = \sum_{j=1}^{n} \sum_{i=1}^{r} \left(\frac{\partial L}{\partial A_{ij}'} \delta A_{ij}' + \frac{\partial L}{\partial A_{ij}''} \partial A_{ij}'' \right) \tag{1.4}$$

Inserting (1.3) and (1.4) into (1.2) and dividing the resulting expression by τ, we have

$$\sum_{j=1}^{n} \sum_{i=1}^{r} \left[\left(\frac{\partial \langle L \rangle_\tau}{\partial A_{ij}'} + \langle Q_j \sin \omega_i t \rangle_\tau \right) \delta A_{ij}' + \left(\frac{\partial \langle L \rangle_\tau}{\partial A_{ij}''} \right) + \right.$$

$$\left. + \langle Q_j \cos \omega_i t \rangle_\tau)\, \delta A_{ij}'' \right] = \frac{1}{\tau} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}}\, \delta q_j \Big|_{t=0}^{t=\tau} \tag{1.5}$$

$$\langle R(t) \rangle_\tau = \frac{1}{\tau} \int_0 R(T)\, dt \tag{1.6}$$

Let us now assume that a function $W(t)$ exists, which satisfies the relations

$$\partial W / \partial q_j^{\cdot} = -\dot{Q}_j (q_\nu, q_\nu^{\cdot}, t) \qquad (j = 1, \ldots, n) \tag{1.7}$$

In this case we can reduce the computation of the mean values of $Q_j \sin\omega_j t$ and $Q_j \cos \omega_l t$, to the process of obtaining the mean of a single function $W(t)$. Indeed, putting

$$q_j^{\cdot}(t) = \sum_{i=1}^{r} (V_{ij}' \cos \omega_i t - V_{ij}'' \sin \omega_i t) \qquad (V_{ij}'(A_{ij}') = \omega_i A_{ij}', V_{ij}''(A_{ij}'') = \omega_i A_{ij}'') \tag{1.8}$$

we find that

$$\frac{\partial \langle W \rangle_\tau}{\partial V_{ij}'} = \left\langle \frac{\partial W}{\partial q_j^{\cdot}} \frac{\partial q_j^{\cdot}}{\partial V_{ij}'} \right\rangle_\tau = -\langle Q_j \cos \omega_i t \rangle_\tau \tag{1.9}$$

$$\frac{\partial \langle W \rangle_\tau}{\partial V_{ij}''} = \left\langle \frac{\partial W}{\partial q_j^{\cdot}} \frac{\partial q_j^{\cdot}}{\partial V_{ij}''} \right\rangle_\tau = \langle Q_j \sin \omega_i t \rangle_\tau \tag{1.10}$$

which make it possible to write (1.5) in the form

$$\sum_{j=1}^{n} \sum_{i=1}^{r} \left[\left(\frac{\partial \langle L \rangle_\tau}{\partial A_{ij}'} + \frac{\partial \langle W \rangle_\tau}{\partial V_{ij}''} \right) \delta A_{ij}' + \left(\frac{\partial \langle L \rangle_\tau}{\partial A_{ij}''} - \frac{\partial \langle W \rangle_\tau}{\partial V_{ij}'} \right) \delta A_{ij}'' \right] = \frac{1}{\tau} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j^{\cdot}}\, \delta q_j \Big|_{t=0}^{t=\tau} \tag{1.11}$$

Taking (1.1) and (1.8) into account we can easily show that $L(q_\nu^{\cdot}, q_\nu)$ and $W(q_\nu, q_\nu^{\cdot}, t)$ can be represented by

$$R\ (x_1, \ldots, x_l) = R\ (k_1, t, \ldots, k_l t) \tag{1.12}$$

periodic with the period equal to 2π in the variables $x_\nu = k_\nu t$ ($\nu = 1, \ldots, l$) and possessing real independent (*) coefficients $k_1, \ldots, k_l$. We shall assume that functions thus obtained

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*) Values $k_1, \ldots, k_l$ shall be called linearly independent, if the equalities $n_1 k_1 + \ldots + n_l k_l = 0$ cannot be fulfilled for any set of integers $(n_1, \ldots, n_l) \neq (0, \ldots, 0)$.

from L and W are Riemann integrable functions of $x_1,\ldots, x_l$. Then, from the theory of uniform distribution [6] it follows that the concept of mean value over infinite time is meaningful for these functions and that it can be computed by means of the following Expression:

$$\langle R(t)\rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau R(k_1 t, \ldots, k_l t)\, dt = \frac{1}{(2\pi)^l}\int_0^{2\pi}\ldots\int_0^{2\pi} R(x_1, \ldots, x_l)\, dx_1, \ldots, dx_l \qquad (1.13)$$

Let us now go to the limit in (1.11) as $T\to\infty$. By the previous assumptions the left-hand side will contain mean values of L and W, while the right-hand side will become zero since the generalized impulses $\partial L/\partial q_j^{\cdot}$ and variations δq_j are bounded. Thus we have

$$\sum_{j=1}^{n}\sum_{i=1}^{r}\left[\left(\frac{\partial\langle L\rangle}{\partial A_{ij}'}+\frac{\partial\langle W\rangle}{\partial V_{ij}''}\right)\delta A_{ij}'+\left(\frac{\partial\langle L\rangle}{\partial A_{ij}''}-\frac{\partial\langle W\rangle}{\partial V_{ij}'}\right)\delta A_{ij}''\right]=0 \qquad (1.14)$$

When a periodic solution is required, we find that the most suitable value of T is that, equal to the corresponding period.

Then, the right-hand side of (1.11) becomes zero since $\partial L/\partial q_j^{\cdot}$ and $\delta q_j^{\cdot}$ are periodic and when we retain the notation $\langle R\rangle$ for the mean value of $R(t)$ over one period, we find that we again arrive at (1.14).

Since the variations $\delta A_{ij}'$ and $\delta A_{ij}''$ are arbitrary, then (1.14) yields a system of $2nr$ equations necessary for the determination of unknown parameters of (1.1)

$$\frac{\partial\langle L\rangle}{\partial A_{ij}'}+\frac{\partial\langle W\rangle}{\partial V_{ij}''}=0, \quad \frac{\partial\langle L\rangle}{\partial A_{ij}''}-\frac{\partial\langle W\rangle}{\partial V_{ij}'}=0 \qquad \begin{pmatrix} i=1,\ldots,r\\ j=1,\ldots,n\end{pmatrix} \qquad (1.15)$$

Thus, we have constructed the required equations by computing the mean values of only two functions, namely L and W. The computation was made even easier by the fact that, as a rule, these functions are symmetric in $q_1,\ldots, q_n$ and $q_1^{\cdot},\ldots, q_n^{\cdot}$. This fact becomes of some importance when polyharmonic oscillations are studied in systems where restoring or resisting forces are given in terms of nonanalytic functions such as $\operatorname{sgn} x$, $x^2\operatorname{sgn} x$ e.a.

Let us now consider an important case. Let

$$Q_j = f_j(q_\nu^{\cdot}) + H_j(t) \qquad (j=1,\ldots,n) \qquad (1.16)$$

where the generalized force

$$H_j(t)=\sum_{i=1}^{m}(H_{ij}'\sin\omega_i t + H_{ij}''\cos\omega_i t) \qquad (m\leqslant r) \qquad (1.17)$$

corresponds to the driving forces and $f_j(q_\nu^{\cdot})$ to resisting forces. Taking into account (1.16), let us now put $W=\Phi - N$ where according to (1.7)

$$N = H_1 q_1^{\cdot} + \ldots + H_n q_n \qquad (1.18)$$

and Φ is the dissipation function defined by [7]

$$\partial\Phi / \partial q_j^{\cdot} = -f_j(q_\nu^{\cdot}) \quad (j=1,\ldots,n) \qquad (1.19)$$

Averaging W we find, that $\langle W\rangle = \langle\Phi\rangle - \langle N\rangle$, where by (1.8) and (1.17)

$$\langle N\rangle = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m}(H_{ij}''V_{ij}' - H_{ij}'V_{ij}'') \qquad (1.20)$$

Inserting $\langle W\rangle$ into (1.15) we arrive at the Eqs. ($H_{ij}'=H_{ij}''=0$ when $i>m$)

$$\frac{\partial\langle L\rangle}{\partial A_{ij}'}+\frac{\partial\langle\Phi\rangle}{\partial V_{ij}''}=-\frac{H_{ij}'}{2}, \qquad \frac{\partial\langle L\rangle}{\partial A_{ij}''}-\frac{\partial\langle\Phi\rangle}{\partial V_{ij}'}=-\frac{H_{ij}''}{2} \qquad \begin{pmatrix} i=1,\ldots,r\\ j=1,\ldots,n\end{pmatrix} \quad (1.21)$$

In a number of cases the following form of a solution is more convenient

$$q_j=\sum_{i=1}^{r} A_{ij}\sin(\omega_i t-\varphi_{ij}), \qquad q_j^{\cdot}=\sum_{i=1}^{r} V_{ij}\cos(\omega_i t-\varphi_{ij})\,(V_{ij}(A_{ij})=\omega_i A_{ij}) \quad (1.22)$$

Converting the driving force (1.17) into an analogous form

$$H_j(t) = \sum_{i=1}^{m} H_{ij} \sin(\omega_i t - \psi_{ij})$$

and using simple operations we obtain, from (1.21),

$$\begin{gathered} -\frac{\partial \langle L \rangle}{\partial A_{ij}} + \frac{1}{V_{ij}} \frac{\partial \langle \Phi \rangle}{\partial \varphi_{ij}} = \frac{1}{2} H_{ij} \cos(\varphi_{ij} - \psi_{ij}) \\ \frac{1}{A_{ij}} \frac{\partial \langle L \rangle}{\partial \varphi_{ij}} + \frac{\partial \langle \Phi \rangle}{\partial V_{ij}} = \frac{1}{2} H_{ij} \sin(\varphi_{ij} - \psi_{ij}) \qquad (i = 1, \ldots, r, j = 1, \ldots, n) \end{gathered} \tag{1.23}$$

2. We shall now consider the oscillations of a linear elastic system with n degrees of freedom. Its point C moves along the x-axis and a periodic force is applied to it. Projection of this force on the x-axis is

$$H(t) = \sum_{i=1}^{m} H_i \sin(ikt - \psi_i), \qquad H_i \geqslant 0 \tag{2.1}$$

In order to limit the amplitudes of oscillations we shall apply, at the same point C, a resisting force

$$f(x^{\cdot}) = -\beta_0 \operatorname{sgn} x^{\cdot} - \beta_1 x^{\cdot} \tag{2.2}$$

where β_0 denotes dry friction while β_1 is the coefficient of viscous friction. We shall assume that the set of natural frequencies Ω_j of the system contains two resonance frequencies $\Omega_1 = n_1 k$ and $\Omega_2 = n_2 k$ where n_1 and $n_2 < m$ are integers, and we shall next discuss the character of the interaction of the corresponding resonant oscillations. (The influence of high frequency perturbations on resonant oscillations of a system with dry friction, was studied in [8]).

Let q_j be the principal coordinates corresponding to the natural frequencies Ω_j. We shall seek the solution of our problem in the form

$$q_j = A_j \sin(n_j kt - \varphi_j) \quad (j = 1, 2), \qquad q_j = 0 \quad (j = 3, \ldots, n) \tag{2.3}$$

which corresponds to a translation of the point C according to

$$x = \sum_{j=1}^{n} \alpha_j(C)\, q_j = \sum_{j=1}^{2} a_j \sin(n_j kt - \varphi_j), \qquad a_j = \alpha_j(C)\, A_j \tag{2.4}$$

Here $\alpha_j(C)$ denote the value of the coefficients of the form of the natural frequencies of the system at C. Moreover we assume, 1° – that the amplitudes of the forced oscillations whose frequencies differ from $n_1 k$ and $n_2 k$ are vanishingly small, 2° – that the forms of resonance oscillations sufficiently resemble the forms of natural oscillations of corresponding frequencies and, consequently, that the solution (2.3) satisfies free oscillation equations (Φ = const)

$$\partial\langle L \rangle / \partial A_j = 0, \qquad \partial\langle L \rangle / \partial\varphi_j = 0 \qquad (j = 1,2) \tag{2.5}$$

Calculating the elementary work done by the nonconservative forces over the possible displacements, we obtain the following expressions for the generalized forces

$$Q_j = f_j(q_\nu^{\cdot}) + H_j(t) \qquad (f_j(q_\nu^{\cdot}) = \alpha_j(C)\, f(x^{\cdot}),\ H_j(t) = \alpha_j(C)\, H)(t)$$

Utilising the results obtained at the end of Section 1 and taking into account (2.5), we have

$$\frac{1}{V_j} \frac{\partial \langle \Phi \rangle}{\partial \varphi_j} = \frac{1}{2} H_j \cos(\varphi_j - \psi_j), \qquad \frac{\partial \langle \Phi \rangle}{\partial V_j} = \frac{1}{2} H_j \sin(\varphi_j - \psi_j) \qquad (j = 1, 2) \tag{2.6}$$

$$V = n_j k a_j, \qquad \langle \Phi \rangle = \beta_0 \langle | x^{\cdot} | \rangle + {}^1/_4 \beta_1 (V_1^2 + V_2^2) \tag{2.7}$$

$$(H_{nj} = H_j, \ \psi_{nj} = \psi_j)$$

where $\langle \Phi \rangle$ is the mean value of the dissipation function

$$\Phi = \beta_0 \, | x^{\cdot} | + {}^1/_2 \beta_1 x^{\cdot 2} \qquad (x^{\cdot} = V_1 \cos (n_1 kt - \varphi_1) + V_2 \cos (n_2 kt - \varphi_2)) \tag{2.8}$$

The magnitude $\langle \Phi_0 \rangle = \beta_0 \langle |\dot{x}| \rangle$ has no unique analytic expression which would be independent of the interrelations between V_1 and V_2 and between n_1 and n_2 in (2.8).

Let us consider the case when $V_1 / V_2 = \gamma$ and $V_2 \geqslant 0$ and let

$$| \gamma | \leqslant 1, \qquad n_1 < n_2 \tag{2.9}$$

When (2.9) holds, expressions in powers of γ can be obtained for $\langle \Phi_0 \rangle$ and the form of these expansions will, in general, depend on the ratios of frequencies and phases in (2.8). These series are difficult to derive, and we give now one of the possible methods of derivation. We write $\langle \Phi_0 \rangle$ as

$$\langle \Phi_0 \rangle = \beta_0 V_2 \langle | \gamma \cos (n_1 kt - \varphi_1) + \cos (n_2 kt - \varphi_2) | \rangle$$

and represent the jodulus in the binomial form

$$| B | = [1 - (1 - B^2)]^{1/2} = 1 + \sum_{n=1}^{\infty} (-1)^n C_{1/2}^n (1 - B^2)^n$$

Next we take its mean value term by term and thus obtain the required series which converge, together with their first derivatives, when $|\gamma| \leqslant 1$. Their first terms are: 1)

$$1) \qquad \langle \Phi_{02} \rangle = 2\pi^{-1} \beta_0 V_2 \, [1 + {}^1/_4 \gamma^2 + {}^1/_{64} \gamma^4 (1 + \cos (\varphi_2 - 2\varphi_1)) + \ldots] \tag{2.10}$$

when $n_2 = 2n_1$,

$$2) \qquad \langle \Phi_{03} \rangle = 2\pi^{-1} \beta_0 V_2 \, [1 + {}^1/_4 \gamma^2 - {}^1/_{24} \gamma^3 \cos (\varphi_2 - 3\varphi_1) + {}^1/_{64} \gamma^4 - \ldots] \tag{2.11}$$

when $n_2 = 3n_1$, and

$$3) \qquad \langle \Phi_0 \rangle = 2\pi^{-1} \beta_0 V_2 \, [1 + {}^1/_4 \gamma^2 + {}^1/_{64} \gamma^4 + \ldots] \tag{2.12}$$

when $n_2 / n_1 = q/p$ (here p and q are numbers, simple with respect to each other, and, when $p = 1$, $q \geqslant 4$ while when $q > p \geqslant 2$, $q \geqslant 3$).

In the case (3) $\langle \Phi_0 \rangle$ also depends on the phase shift (coefficients containing the phase terms correspond to the powers of γ higher than fourth), but the dependence is quite weak and can be neglected.

Let us consider the case (2) in more detail. Inserting (2.11) into (2.6), discarding some inessential terms, and taking $\varphi_2 - 3\varphi_1 = \delta$ we obtain

$$-{}^1/_2 \beta_0 \pi^{-1} \gamma^2 \sin \delta = H_1 \cos (\varphi_1 - \psi_1)$$

$$\beta_1 V_1 + 2\beta_0 \pi^{-1} V_1 V_2^{-1} (1 - {}^1/_4 \gamma \cos \delta) = H_1 \sin (\varphi_1 - \psi_1)$$

$${}^1/_6 \beta_0 \pi^{-1} \gamma^3 \sin \delta = H_2 \cos (\varphi_2 - \psi_2)$$

$$\beta_2 V_2 + 4\beta_0 \pi^{-1} (1 - {}^1/_4 \gamma^2 + {}^1/_{12} \gamma^3 \cos \delta) = H_2 \sin (\varphi_2 - \psi_2) \tag{2.13}$$

Then, combining the first equation of (2.13) with the second and third with the fourth, we obtain

$$| \operatorname{tg} (\varphi_1 - \psi_1) | = \left| \frac{2\pi \beta_1 V_2}{\beta_0 \gamma \sin \delta} + \frac{4 (1 - {}^1/_4 \gamma \cos \delta)}{\gamma \sin \delta} \right| > 4, \qquad \text{or} \qquad \sin (\varphi_1 - \psi_1) > 0.97$$

$$| \operatorname{tg} (\varphi_2 - \psi_2) | = \left| \frac{6\pi \beta_1 V_2}{\beta_0 \gamma^3 \sin \delta} + \frac{24 (1 - {}^1/_4 \gamma^2 + {}^1/_{12} \gamma^3 \cos \delta)}{\gamma^3 \sin \delta} \right| > 18, \qquad \text{or} \qquad \sin (\varphi_2 - \psi_2) > 0.99$$

Putting the values of the sines as equal to unity, we obtain from the second and fourth Eqs. of (2.13)

$$V_1=\frac{H_1}{\beta_1}\left[1+\frac{2\beta_0}{\pi\beta_1V_2}\left(1-\frac{\gamma}{4}\cos\delta\right)\right]^{-1}\qquad V_2=\frac{H_2}{\beta_1}\left[1-\frac{4\beta_0}{\pi H_2}\left(1-\frac{\gamma^2}{4}+\frac{\gamma^3}{12}\cos\delta\right)\right] \tag{2.14}$$

which are convenient for use in the method of successive approximations. Second step of this method yields approximate formulas for resonance amplitudes

$$a_1=\frac{H_1(1-b_2)}{\beta_1n_1k}\left[1-\frac{b_2}{2}-\frac{b_2H_1\cos\delta}{4H_2(2-b_2)}\right]^{-1} \tag{2.15}$$

$$a_2=\frac{H_2}{\beta_1n_2k}\left[1-b_2+\frac{b_2H_1^2}{H_2^2(2-b_2)^2}-\frac{2b_2H_1^3\cos\delta}{3H_2^3(2-b_2)^3}\right]\qquad\left(0<b_2=\frac{4\beta_0}{\pi H_2}<1\right)$$

from which we see that the resonance amplitudes are strongly influenced by the phase shift δ. For example, when $b_2\approx 1$ and $H_2\approx H_1+0.6\,\beta_0$, then the value of deviation from the mean may reach 25% for a_1 and 30% for a_2.

When the frequencies are less restricted and are simply $n_1<n_2$, we put $\gamma\leqslant 0.5$ and, obtain, from (2.10) to (2.12),

$$\langle\Phi_0\rangle=2\pi^{-1}\beta_0V_2\,(1+{}^1/_4\gamma^{-2})$$

with sufficient accuracy.

Now, (2.6) yield

$$\cos(\varphi_j-\psi_j)=0,\qquad\text{or}\qquad\varphi_j-\psi_j={}^1/_2\pi\qquad(j=1,2)$$
$$\beta_1V_1+2\beta_0V_1/\pi V_2=H_1,\qquad\beta_1V_2+4\beta_0\pi^{-1}(1-{}^1/_4\gamma^2)=H_2 \tag{2.16}$$

which in turn yield approximate formulas for resonance amplitudes

$$a_1=\frac{H_1(1-b_2)}{\beta_1n_1k\,(1-b_2/2)},\qquad a_2=\frac{H_2}{\beta_1n_2k}\left[1-b_2+\frac{b_2H_1^2}{H_2^2(2-b_2)^2}\right] \tag{2.17}$$

and a condition in which $|\gamma|\leqslant 0.5$ and

$$H_2\geqslant 2H_1+0.56\,\beta_0 \tag{2.18}$$

To estimate the interaction between the harmonics themselves, we must find the amplitudes a_{j0} of the resonant oscillations occurring in the system under consideration when the forces $H_j\sin(n_jkt-\psi_j)$ act on it separately.

If the solution is sought in the form

$$x_{j0}(t)=a_{j0}\sin(n_jkt-\varphi_j)$$

then we can easily show that in this case

$$a_{j0}=\frac{H_j(1-b_j)}{\beta_1n_jk},\qquad b_j=\frac{4\beta_0}{\pi H_j}\leqslant 1;\qquad a_{j0}=0,\qquad b_j>1$$

Comparing a_j with a_{j0} and taking into account the fact that, by (2.18) $b_1>b_2$, we find that when $b_1<1$,

$$\frac{a_1}{a_{10}}=\frac{1-b_2}{(1-b_2/2)(1-b_1)}>1,\qquad\frac{a_2}{a_{20}}=1+\frac{b_2H_1^2}{H_2^2(1-b_2)(2-b_2)^2}\geqslant 1 \tag{2.19}$$

Thus, the interaction of harmonics in the presence of dry friction leads to an increase in the values of both resonance amplitudes, and the 'slower' harmonic (the harmonic with a smaller velocity amplitude) may exhibit larger variation in amplitude. This can easily be seen from (2.19).

We note another fact. From (2.17) we see that the resonance amplitude a_1 is a linear function of the amplitude of the force H_1. Since this feature is inherent in the systems with viscous (linear) friction, a linearization of dry friction is effected by the harmonic of higher velocity amplitude and frequency, on the 'slower' harmonic. It should be stressed that the expression 'higher frequency' does not imply large frequency differences (see [8]). It will be sufficient for the frequency ratio to be 6/5, 3/2, 2, 3, etc. Moreover, it is not at all necessary for the linearization harmonic to have a higher frequency, it is only necessary that its velocity amplitude is sufficiently large as compared with the velocity amplitude of the other harmonic.

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